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On Phi-Pseudo-Monotonicity and Approximation-Solvability of Nonlinear Equations

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We generalize the results on the approximation-solvability of nonlinear functional equations to the case of ϕ-pseudo-monotone mappings—a new and rather general class of mappings. An application is considered. © 1994 Academic Press, Inc.

1. INTRODUCTION

Petryshyn [2] considered an approximation scheme and applied it to the general approximation-solvability of nonlinear equations in general normed spaces. This scheme is specifically more application-oriented toward the numerical methods in the sense that solutions of an equation in some function space X can be approximated by solutions of equations in some discrete space, say $X_n = \mathbf{R}^n$. As an example, we can consider an integral equation

$$x(t) = \int_0^1 k(t, s)x(s)\, ds + f(t) \tag{1.1}$$

on $C([0, 1])$ with $k \in C([0, 1] \times [0, 1])$, and apply the convergent mechanical quadrature to approximate $\int_0^1 y(s)\, ds$, that is, we write

$$\int_0^1 y(s)\, ds = \sum_{j=1}^{n} a_{nj} y(s_{nj}) + r_n(y) \tag{1.2}$$

with coefficients $a_{nj} \geq 0$, interpolation points $0 \leq s_{n1} < \cdots < s_{nn} \leq 1$ and the remainder r_n such that $r_n(y) \to 0$ on $C([0, 1])$. Thus, the integral

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equation is approximated by the discrete system

$$c_{ni} = \sum_{j=1}^{n} a_{nj}\, k(s_{ni}, s_{nj}) c_{nj} + f(s_{ni}) \qquad \text{for } i = 1, \ldots, n, \tag{1.3}$$

obtained by neglecting r_n and taking $t = s_{ni}$ for $i = 1, \ldots, n$.

The purpose of this paper is twofold. First, we generalize the results on approximation-solvability, by applying the theorem of Petryshyn [2] to the case of ϕ-pseudo-monotone mappings—a new class of mappings in the sense of Vainberg [4]. Second, we give an application to the numerical range—a generalization of the Zarantonello numerical range [7].

DEFINITION 1.1. Let X and Y be complex Banach spaces, and $\phi: X \to Y$ be such that

(i) $\phi(X)$ is dense in Y, and
(ii) for each $x \in X$ and each $t \geq 0$, $\phi(tx) = t\phi(x)$.

A mapping $A: X \to Y^*$ from X to Y^*, the dual of Y, is said to be ϕ-pseudo-monotone if there exists a constant $d > 0$ such that

$$|[Ax - Au, \phi(x-u)]| \geq d\|x-u\|\,\|\phi(x-u)\| \qquad \text{for all } x, u \in X, \tag{1.4}$$

where $[\cdot, \cdot]$ is the pairing between Y^* and Y.

Consequently,

$$|[Ax - Au, \phi(x-u)]| \geq d\big|\,\|x\| - \|u\|\,\big|\,\|\phi(x-u)\| \qquad \text{for all } x, u \in X. \tag{1.5}$$

A is pseudo-monotone in the sense of Vainberg [4] when $Y = X$ (reflexive) and $\phi = I$.

However, usually pseudo-monotonicity carries a different meaning [8].

2. APPROXIMATION-SOLVABILITY

Suppose X is a separable reflexive complex Banach space with $\dim X = \infty$, and $A: X \to X^*$ from X to its dual X^* is ϕ-pseudo-monotone and continuous. Let (X_n) be a Galerkin scheme in X such that

$$X_n = \operatorname{span}(e_{1n}, \ldots, e_{n'n}), \qquad n = 1, 2, \ldots \tag{2.1}$$

Let $E_n: X_n \to X$ be the embedding mapping corresponding to $X_n \subseteq X$. We construct $R_n: X \to X_n$ in such a manner that, for each $x \in X$, there exists at least an element $R_n x$ such that

$$\|x - R_n x\| = \operatorname{dist}(x, X_n). \tag{2.2}$$

We consider the equation

$$Ax = b, \qquad x \in X, b \in X^*, \tag{2.3}$$

along with approximate equations

$$E_n^* A E_n x_n = E_n^* b, \qquad x_n \in X_n, n = 1, 2, \ldots, \tag{2.4}$$

with respect to an approximation scheme $\pi = \{X_n, \phi_n, E_n, R_n, X_n^*, E_n^*\}$, represented by a diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] & X \arrow[r, "A"] \arrow[d, shift right, "R_n"'] & X^* \arrow[d, "E_n^*"] \\ X_n \arrow[u, "E_n"] \arrow[r, "\phi_n"] & X_n \arrow[r, "A_n"] \arrow[u, shift right, "E_n"'] & X_n^* \end{tikzcd} \tag{2.5}$$

where $A_n = E_n^* A E_n$ and $\phi_n = R_n \phi E_n$ are continuous. For more details on approximation-solvability, see Deimling [1], Petryshyn [2], Rosen [3], and Zeidler [8].

For $n = 1, 2, \ldots$, approximate equations (2.4) are equivalent to Galerkin equations

$$[Ax_n, e_{jn}] = [b, e_{jn}] \tag{2.6}$$

for $x_n \in X_n, j = 1, \ldots, n'$.

Here and in what follows, operators $E_n\colon X_n \to X$ and $E_n^*\colon X^* \to X_n^*$ are linear and continuous. The symbols "$\to$" and "$\xrightarrow{w}$" shall denote the strong convergence and weak convergence, respectively. Also, $\mathbf{K}$ denotes the real or complex field.

We recall some definitions for the sake of the completeness.

Definition 2.1 (Compatibility). The approximation scheme $\pi = \{X_n, \phi_n, E_n, R_n, X_n^*, E_n^*\}$ is called *compatible* if

$$\lim_{n\to\infty} \|E_n R_n x - x\|_X = 0 \qquad \text{for all } x \in X. \tag{2.7}$$

Definition 2.2 (Admissible Inner Approximation Scheme). The approximation scheme π is said to be an *admissible inner approximation scheme* iff

(A1) X and X^* are infinite-dimensional normed spaces over $\mathbf{K}$.

(A2) X_n and X_n^* are normed spaces over $\mathbf{K}$ with $\dim X_n = \dim X_n^* < \infty$ for all n.

(A3) E_n: $X_n \to X$ and E_n^*: $X^* \to X_n^*$ are linear and continuous with $\sup\|E_n\| < \infty$ and $\sup\|E_n^*\| < \infty$.

(A4) The compatibility condition is satisfied.

DEFINITION 2.3 (Solvability). The equation $Ax = b$ is said to be *solvable* if it has a solution for each $b \in X^*$.

DEFINITION 2.4 (Unique Approximation-Solvability). The equation $Ax = b$ is called uniquely *approximation-solvable* if, for each $b \in X^*$, the following hold:

(i) $Ax = b$, $x \in X$, has a unique solution.

(ii) For each $n \geq n_0$, the approximate equation

$$E_n^* A E_n x_n = E_n^* b, \qquad x_n \in X_n, \tag{2.8}$$

has a unique solution.

(iii) The sequence (x_n) converges to the solution x of the equation $Ax = b$ in the sense that

$$\lim_{n\to\infty}\|E_n x_n - x\|_X = 0. \tag{2.9}$$

DEFINITION 2.5 (A-Properness). The operator A: $X \to X^*$ is *approximation-proper* (abbreviated as *A-proper*) with respect to the approximation scheme π if the following holds:

Let (n') be any subsequence of the sequence of natural numbers. If $(x_{n'})$ is a sequence with $x_{n'} \in X_{n'}$ for all n', and if

$$\lim_{n\to\infty}\|A_{n'}x_{n'} - E_{n'}^* b\|_{X_{n'}^*} = 0 \text{ for fixed } b \in X^* \qquad \text{and} \qquad \sup\|x_{n'}\| < \infty,$$
(2.10a), (2.10b)

then there exists a subsequence $(x_{n''})$ such that

$$\lim_{n\to\infty}\|E_{n''}x_{n''} - x\|_X = 0 \qquad \text{and} \qquad Ax = b. \quad \text{(2.11a), (2.11b)}$$

DEFINITION 2.6 (Consistency). The approximation scheme π is called *consistent* if, for all $x \in X$,

$$\lim_{n\to\infty}\|E_n^* Ax - A_n R_n x\|_{X_n^*} = 0. \tag{2.12}$$

DEFINITION 2.7 (Stability). The approximation scheme π is said to be *stable* if there exists an n_0 such that

$$\|A_n x - A_n u\|_{X_n^*} \geq d\|x - u\|_{X_n} \tag{2.13}$$

for all $x, u \in X_n$ and $n \geq n_0$, where $d > 0$.

Before proceeding to the main result, we need to state the crucial theorem of Petryshyn [2] on the inner approximation-solvability.

LEMMA 2.1. *If the approximation scheme $\pi = \{X_n, \phi_n, E_n, R_n, X_n^*, E_n^*\}$ is an admissible inner approximation scheme with consistency and stability, then the equation*

$$Ax = b, \qquad x \in X, \tag{2.14}$$

is uniquely approximation-solvable for each $b \in X^*$ iff A is A-proper.

Now we establish the main result on the approximation-solvability of nonlinear equations involving ϕ-pseudo-monotone operators.

THEOREM 2.1. *Let X be a separable reflexive complex Banach space with* $\dim X = \infty$. *Let $\pi = \{X_n, \phi_n, E_n, R_n, X_n^*, E_n^*\}$ be an approximation scheme for the pair (X, X^*) represented by the diagram* (2.5), *and let $\phi\colon X \to X$ be continuous such that*

$$R_n \phi E_n x = \phi x \qquad \text{for all } x \in X_n \text{ and each } n. \tag{2.15}$$

If $A\colon X \to X^*$ is ϕ-pseudo-monotone and continuous, then, for each $b \in X^*$, the equation

$$Ax = b, \qquad x \in X, \tag{2.16}$$

is uniquely approximation-solvable.

For $\phi = I$, we arrive at the following.

COROLLARY 2.1. *If X is a separable reflexive complex Banach space with* $\dim X = \infty$, *$\pi_0 = \{X_n, E_n, R_n, X_n^*, E_n^*\}$ is an approximation scheme for the pair (X, X^*), and $A\colon X \to X^*$ is pseudo-monotone and continuous, then, for each $b \in X^*$, the equation*

$$Ax = b, \qquad x \in X, \tag{2.17}$$

is uniquely approximation-solvable.

Proof of Theorem 2.1. As our method of proof is based upon Lemma 2.1, we first show that the approximation scheme $\pi = \{X_n, \phi_n, E_n, R_n, X_n^*,$

$E_n^*\}$ is an admissible inner approximation scheme such that π is consistent and stable. Second, we show that $A: X \to X^*$ is A-proper.

Admissible Inner Approximation Scheme. Since $\|E_n\| = 1$ (and hence $\|E^*\| = 1$) and (X_n) is a Galerkin scheme, $\mathrm{dist}(x, X_n) \to 0$ as $n \to \infty$ for all $x \in X$. This implies that $\|R_n x - x\| \to 0$ as $n \to \infty$, and thus, the compatibility condition is satisfied.

Consistency. Since A is continuous and the compatibility condition (2.7) is satisfied, the consistency condition is as follows: Since

$$\|AE_nR_nx - Ax\| \to 0 \qquad \text{as } n \to \infty \text{ and } \|E_n^*\| = 1, \tag{2.18}$$

this implies that, as $n \to \infty$,

$$\|E_n^*Ax - A_nR_nx\| = \|E_n^*Ax - E_n^*AE_nR_nx\| \leq \|E_n^*\|\,\|Ax - AE_nR_nx\| \to 0. \tag{2.19}$$

Stability. For all $x, y \in X_n$, we obtain

$$\begin{aligned}\|A_nx - A_ny\|\,\|\phi(x - y)\| &\geq |[A_nx - A_ny, \phi(x - y)]| \\ &= |[E_n^*AE_nx - E_n^*AE_ny, \phi(x - y)]| \\ &= |[AE_nx - AE_ny, E_n\phi(x - y)]| \\ &= |[Ax - Ay, \phi(x - y)]| \\ &\geq d\|x - y\|\,\|\phi(x - y)\|,\end{aligned} \tag{2.20}$$

and consequently,

$$\|A_nx - A_ny\| \geq d\|x - y\| \qquad \text{for all } x, y \in X_n. \tag{2.21}$$

A-Properness. Assume $\sup\|x_n\| < \infty$ for all $x_n \in X_n$, and

$$\|A_nx_n - E_n^*b\| = \|E_n^*Ax_n - E_n^*b\| \to 0 \qquad \text{as } n \to \infty \text{ for all } n. \tag{2.22}$$

Since X is reflexive, there exists a subsequence, again denoted by (x_n), such that

$$x_n \xrightarrow{w} x \qquad \text{in } X \text{ as } n \to \infty. \tag{2.23}$$

This means that it suffices to show that

$$x_n \to x \text{ as } n \to \infty \qquad \text{and} \qquad Ax = b. \tag{2.24}$$

Since $\|R_n x - x\| \to 0$ as $n \to \infty$, $x_n \xrightarrow{w} x$ as $n \to \infty$ implies that

$$x_n - R_n x \xrightarrow{w} 0 \qquad \text{as } n \to \infty. \tag{2.25}$$

Thus, as $n \to \infty$, with (2.22) we get

$$\begin{aligned}\|E_n^* A x_n - E_n^* A R_n x\| &= \|(E_n^* A x_n - E_n^* b) + (E_n^* b - E\cdot_n^* A R_n x)\| \\ &\leq \|E_n^* b - E_n^* A x\| + \|E_n^*(b - Ax)\|. \end{aligned}\tag{2.26}$$

It follows from condition (2.20) and the above results that we get, for $x_n \in X_n$ as above, as $n \to \infty$,

$$\begin{aligned} d\|x_n - R_n x\| \, \|\phi(x_n - R_n x)\| &\leq |[A_n x_n - A_n R_n x, \phi(x_n - R_n x)| \\ &= |[E_n^* A x_n - E_n^* A R_n x, \phi(x_n - R_n x)]| \to 0. \end{aligned}\tag{2.27}$$

This also implies that $x_n - R_n x \to 0$ as $n \to \infty$. Since $\sup\|E_n\| < \infty$, we find

$$\|E_n x_n - E_n R_n x\| \to 0 \qquad \text{as } n \to \infty, \tag{2.28}$$

and the compatibility condition gives

$$\|E_n R_n x - x\| \to 0 \qquad \text{as } n \to \infty. \tag{2.29}$$

Thus,

$$\|E_n x_n - x\| \to 0 \qquad \text{as } n \to \infty. \tag{2.30}$$

It only remains to show that $Ax = b$. Since $x_n - R_n x \to 0$ as $n \to \infty$ implies taht $x_n \to x$ as $n \to \infty$, and since A is continuous, we obtain $Ax = b$. Hence A is A-proper.

Now the equation $Ax = b$ is uniquely approximation-solvable by Lemma 2.1, and this completes the proof.

3. Application to Numerical Range

This section deals with the approximation-solvability of the equations involving numerical ranges.

Definition 3.1 (Numerical Range). Let $A\colon X \to X^*$ be a mapping from a separable reflexive complex Banach space X to its dual X^*. The set

$$n_0[A] = \left\{\frac{[Ax, \phi x] + [Ay - Az, \phi(y - z)]}{[Jx, \phi x] + [Jy - Jz, \phi(y - z)]}: x, y, z \in X, y \neq z\right\} \quad (3.1)$$

is called the *numerical range* of A, where $J: X \to X^*$ is strictly monotone normalized duality, and $\phi: X \to X$ is continuous such that

(i) $\phi(X)$ is dense in X; and
(ii) for each $t \geq 0$, $\phi(tx) = t\phi(x)$.

A continuous function $a: R^+ \to R^+$ is called a *gauge function* if $a(0) = 0$, and a is strictly increasing. Let X be a real reflexive Banach space and X^* its dual. A mapping $J: X \to X^*$ is called a *duality mapping* between X and X^* with respect to a gauge function a if

$$[Jxx] = a(\|x\|)\|x\| \text{ and } \|Jx\| = a(\|x\|) \qquad \text{for } x \in X. \quad (3.2)$$

We note that if $a(t) = t$, J is said to be a *normalized duality mapping*. If X^* is strictly convex, then J is uniquely determined by a, and if X is also reflexive, then J is a single-valued semicontinuous mapping of X onto X^*, which is bounded and positively homogenous. Furthermore, J is monotone and satisfies, for all $x, y \in X$,

$$[Jx - Jy, x - y] \geq (a(\|x\|) - a(\|y\|))(\|x\| - \|y\|). \quad (3.3)$$

Also, we have

$$[Jx - Jy, x - y] = [Jx, x - y] - [Jy, x - y] \geq |a(\|x\|) - a(\|y\|)|\,\|x - y\|. \quad (3.4)$$

If J is normalized duality, then

$$[Jx - Jy, x - y] \geq (\|x\| - \|y\|)^2 \qquad \text{for all } x, y \in X, \quad (3.5)$$

which impliex that

$$[Jx - Jy, x - y] \geq |\,\|x\| - \|y\|\,|\,\|x - y\| \qquad \text{for all } x, y \in X. \quad (3.6)$$

Next, we describe some of the properties of the numerical range $n_0[A]$ along with some special cases of significance.

Clearly, $n_0[A]$ reduces to the numerical range (Verma [6]) defined by

$$V[A] = \left\{\frac{\langle Ax, x\rangle + \langle Ay - Az, y - z\rangle}{\|x\|^2 + \|y - z\|^2}: x, y, z \in X, y \neq z\right\}, \quad (3.7)$$

where X is a Hilbert space and ϕ is the identity. Furthermore, $n_0[A]$ coincides with the Zarantonello numerical range [7] defined by

$$\mathcal{N}[A] = \left\{\frac{\langle Ay - Az, y - z\rangle}{\|y - z\|^2}: y, z \in X, y \neq z\right\} \tag{3.8}$$

where X is a Hilbert space with $\phi = I$ and $x = 0$.

THEOREM 3.1. *Let A, B: $X \to X^*$ be mappings from a reflexive Banach space X to its dual X^*, and $\lambda \in \mathbf{K}$. Then*

(i) $n_0[\lambda A] = \lambda n_0[A]$.

(ii) $n_0[A + B] \subset n_0[A] + n_0[B]$.

(iii) $n_0[A - \lambda J] = n_0[A] - \{\lambda\}$,

where J: $X \to X^*$ is strictly monotone normalized duality.

Proof. The proof follows from the Definition 3.1.

We now establish the main result of the section.

THEOREM 3.2. *Let A: $X \to X^*$ be continuous from a separable reflexive complex Banach space X to its dual X^*. If X and X^* are locally uniformly convex, a number $\lambda \in \mathbf{K}$ has a positive distance from the numerical range of A, $n_0[A]$, that is,*

$$d = \inf\{|\lambda - \mu|: \mu \in n_0[A]\} > 0, \tag{3.9}$$

ϕ: $X \to X$ is continuous, and J: $X \to X^*$ is a normalized duality, then the equation

$$Ax - \lambda Jx = b, \qquad x \in X, \tag{3.10}$$

is uniquely approximation-solvable for each $b \in X^*$.

COROLLARY 3.1. *Theorem* 3.2 *reduces to a result of the author* [6, *Theorem* 2.2] *when X is a Hilbert space and ϕ is the identity.*

We derive the following corollary when X is a Hilbert space with $x = 0$ and $\phi = I$.

COROLLARY 3.2. (Zeidler [8, Theorem 34.C]). *Suppose A: $X \to X$ is continuous from a separable Hilbert space X into itself. If a number $\lambda \in K$ has a positive distance from the Zarantonello numerical range of A, $\mathcal{N}[A]$, that is,*

$$d = \inf\{|\lambda - \mu|: \mu \in \mathcal{N}[A]\} > 0, \tag{3.11}$$

then the equation

$$Ax - \lambda Ix = b, \qquad x \in X, \tag{3.12}$$

is uniquely approximation-solvable for each $b \in X$.

Proof of Theorem 3.2. Before we can apply Theorem 2.1, we need to obtain the key estimates. Since $J: X \to X^*$ is strictly monotone, we find, for $x, y, z, \in X$ with $y \neq z$,

$$\begin{aligned}
&|[(A - \lambda J)x, \phi x] + [(A - \lambda J)y - (A - \lambda J)z, \phi(y - z)]| \\
&\quad = |[Ax, \phi x] - \lambda[Jx, \phi x] + [Ay - Az, \phi(y - z)] \\
&\qquad - \lambda[Jy - Jz, \phi(y - z)]| \\
&\quad = |[Ax, \phi x] + [Ay - Az, \phi(y - z)] - \lambda([Jx, \phi x] \\
&\qquad + [Jy - Jz, \phi(y - z)])| \qquad\qquad (3.13) \\
&\quad = \left|\frac{[Ax, \phi x] + [Ay - Az, \phi(y - z)]}{[Jx, \phi x] + [Jy - Jz, \phi(y - z)]} - \lambda\right| |[Jx, \phi x] \\
&\qquad\qquad + [Jy - Jz, \phi(y - z)]| \\
&\quad \geq d \operatorname{Re}([Jx, \phi x] + [Jy - Jz, \phi(y - z)]) \\
&\quad \geq d(\|x\| \, \|\phi x\| + |\, \|y\| - \|z\| \,| \, \|\phi(y - z)\|).
\end{aligned}$$

For $x = 0$, we obtain an estimate

$$|[A - \lambda J)y - (A - \lambda J)z, \phi(y - z)]| \geq d|\, \|y\| - \|z\| \,| \, \|\phi(y - z)\| \tag{3.14}$$

for all $y, z \in X$.

The proof can be obtained from that of Theorem 2.1.

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